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The instability of short waves on a vortex ring

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A simple model for the experimentally observed instability of the vortex ring to azimuthal bending waves of wavelength comparable with the core size is presented. Short-wave instabilities are discussed for both the vortex ring and the vortex pair. Instability for both the ring and the pair is predicted to occur whenever the self-induced rotation of waves on the filament passes through zero. Although this does not occur for the first radial bending mode of a vortex filament, it is shown to be possible for bending modes with a more complex radial structure with at least one node at some radius within the core. The previous work of Widnall & Sullivan (1973) is discussed and their experimental results are compared with the predictions of the analysis presented here.

1. Introduction

The stability of a vortex ring of small cross-section to bending displacements of the filament in waves about the azimuth was investigated theoretically by Widnall & Sullivan (1973, referred to below as I). They also observed experimentally that the vortex ring was unstable to bending waves. Their measurements of both the amplification rate and mode shape of the instability for a given vortex ring were in reasonable agreement with the theoretical predictions.

The theoretical investigation was based on the previous work of Widnall, Bliss & Zalay (1971, hereafter referred to as II) and Bliss (1970), in which a general asymptotic analysis was presented to predict the self-induced motion of vortex filaments of small cross-section. This theory requires that changes along the filament be negligible in comparison with changes over the cross-section of the vortex core. In other words, the wavelength of perturbations along the filament must be large in comparison with the radius of the vortex core.

Unfortunately, the wavelength of the instability for the vortex ring predicted by this theory does not fulfil this requirement. The theoretical predictions would have been disregarded if it had not been for the rather good agreement between theory and experiment on the general features of the instability, the prediction of the number of waves in the unstable mode for given ring properties and the prediction of the amplification rate.

The purpose of this paper is to present a simple model for the vortex-ring instability which can explain current experimental observations and serve as a basis for a more detailed and considerably more complex asymptotic analysis.

On the basis of the arguments presented here, we have concluded that the vortex ring is unstable to bending displacements of the filament but that the mode across the vortex core is a second radial mode, that is, the displacement of the filament changes sign as one moves out from the centre of the core.

2. Formulation

We consider the stability of a slender vortex ring to bending waves around the azimuth under the condition that the wavelength is comparable with the core size. (Slender means that the ratio of the vortex-core radius to the radius of the ring is small; i.e. $a/R \ll 1$.) In this case, we may still construct an asymptotic solution in the limit $a/R \to 0$, but we must allow ka to be of order one, where k is the wavenumber along the filament. Whereas the problem treated in I was the limit $a/R \to 0$, $ka \ll 1$, the present problem is the limit $a/R \to 0$, $ka = O(1)$, so that $kR \to \infty$ as $a/R \to 0$. In this case, many of the simplifying features of the long-wave analysis of vortex filaments cannot be used. For example, the cut-off analysis cannot be used to calculate self-induced motion nor can the vortex be replaced by a single line filament for the purpose of calculating the velocities induced at points along the filament by short-wave perturbations around the ring. Even taking advantage of asymptotic methods in the limit $a/R \to 0$, the analysis that must be done to calculate the stability of the vortex ring under the condition $ka = O(1)$ is much more complex and, as we shall see, requires the solution of several separate problems, some of which have to be done numerically.

In order to motivate this considerably more complex stability calculation, we here consider a simple analysis which demonstrates some essential features of the instability and which will also serve as a basis for understanding a more complete calculation. This is done by considering the vortex-ring instability along lines similar to those for the vortex-pair instability. This model allows us to consider the possibility that the behaviour of the higher radial bending modes is important for the stability of the vortex ring (as well as for the pair). The general features of the instability of both the pair and the ring are shown in figure 1.

The instability of the vortex pair to bending-wave perturbations was first considered by Crow (1970). Since the flow outside the vortex filaments is potential, the stability calculation can be done by considering the motion of the filaments that results from the perturbations. The vortex filaments move with a velocity that is a combination of the velocity of the filament that results from its perturbation in the non-uniform steady flow field of the other filament, the velocity induced by the displacements of the other filament and the self-induced rotation Ω of the sinusoidally perturbed, straight vortex. Instability of the vortex pair to a perturbation of wavenumber k occurs whenever the tangential component of the velocity induced at the filament both by the presence and perturbations of the other filament can balance the self-induced rotation Ω of the filament itself. Under these conditions, the perturbation will increase in amplitude as the filament diverges in the induced velocity field. Typical stability calculations for the vortex pair indicate two ranges of unstable waves: long waves with $\lambda/b \approx 8$ and short waves with $\lambda/b < 1$, where b is the separation between the vortex filaments.

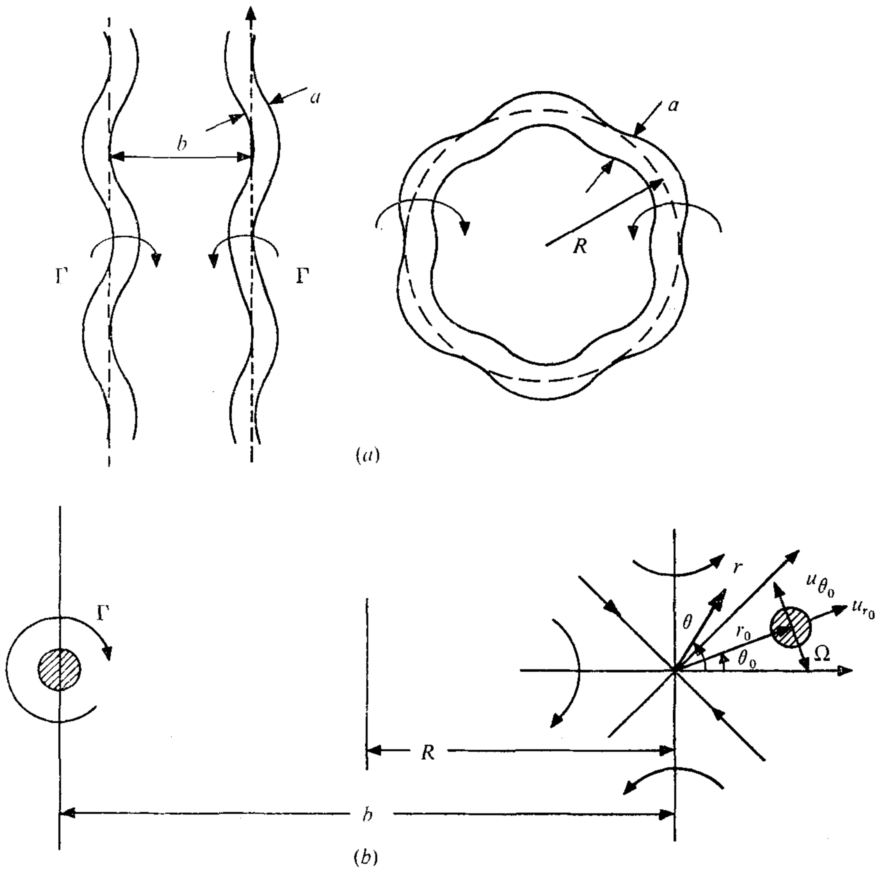

FIGURE 1. Vortex-pair and vortex-ring instability. (*a*) General features of the instability. (*b*) View of the cross-section of the filaments.

For the short waves, it is the presence of the neighbouring filament rather than the perturbations on that filament that plays the dominant role in the instability. The presence of the other filament produces a stagnation-point flow in the neighbourhood of the vortex, whereas the velocity induced at the vortex owing to short-wave perturbations of the other filament is negligible. In the local cylindrical (r, θ) co-ordinate system centred on the unperturbed position of the vortex filament as shown in figure 1, this stagnation-point flow is given by

$$u_r = \frac{\Gamma}{2\pi b^2} r \sin 2\theta, \quad u_\theta = \frac{\Gamma}{2\pi b^2} r \cos 2\theta, \qquad (1a, b)$$

where u_r is the radial velocity and u_θ is the tangential velocity. The velocity of the vortex cross-section that results from a displacement in the θ_0 direction of magnitude r_0 is the sum of the radial and tangential velocities at its position (r_0, θ_0) in the stagnation-point flow (1), given by

$$u_{r_0} = \frac{\Gamma}{2\pi b^2} r_0 \sin 2\theta_0, \quad u_{\theta_0} = \frac{\Gamma}{2\pi b^2} r_0 \cos 2\theta_0, \qquad (2a, b)$$

plus the self-induced rotation Ω of a sinusoidally perturbed filament.† Whenever u_{θ_0} is equal and opposite to $r_0\Omega$, the vortex will diverge along $\theta = \theta_0$ (as sketched in figure 1) with velocity u_{r_0} and the position r_0 of the vortex will increase exponentially in time with a non-dimensional amplification rate $\overline{\alpha} = \alpha/(\Gamma/2\pi b^2)$ given by

$$\overline{\alpha} = \sin 2\theta_0. \tag{3}$$

(The complete vortex-pair stability calculation gives $\overline{\alpha} \cong 0{\cdot}8$ and θ_0 somewhat greater than 45° for the most unstable long waves while for the short waves $\overline{\alpha} = 1$ and $\theta_0 \cong 45°$.)

The calculations of the vortex-pair instability (Crow 1970; II) make use of the asymptotic formula for the self-induced rotation rate for long waves on a straight vortex filament:

$$\Omega = -\frac{\Gamma}{4\pi}k^2\left[\ln\frac{1}{ka_e}+\tfrac{1}{4}+\ln 2-\gamma\right], \tag{4}$$

where a_e is the effective core size of the vortex. The analysis of II demonstrates that a vortex with a given core size and distribution of vorticity is kinematically equivalent to a vortex filament of the same circulation, with constant vorticity and of a core size a_e as regards its self-induced motion for *long-wave* disturbances.

For vortex filaments without axial flow, the short-wave instability obtained in these calculations is thought to be spurious since it is predicted to occur at a wavenumber for which the asymptotic result (4) for self-induced motion is not valid because $ka_e = O(1)$. The variation of the asymptotic result with ka_e is shown in figure 2. This expression predicts first an increase then a decrease in Ω with increasing ka_e. At $ka_e = 1{\cdot}44$, Ω is predicted to be zero. Dispersion relations for the first radial mode of bending waves on a vortex filament do not in reality behave in this way. As an example, the exact result for a vortex with constant vorticity, obtained numerically from the dispersion relation presented by Moore & Saffman (1972), is shown in figure 2 for comparison with the asymptotic result.

If we consider the condition for instability $r_0\Omega = u_{\theta_0}$, we can see from (2) that, if Ω is predicted to be zero for some ka_e, the vortex will position itself at $\theta_0 = 45°$ and diverge with a velocity u_{r_0} corresponding to $\overline{\alpha} = 1$. This is essentially the short-wave mode for the vortex pair identified by Crow (1970) and others. Since the actual dispersion relation for bending modes of this type on vortex filaments (as shown in figure 2) does not take the value $\Omega = 0$ for any value of ka_e this is apparently a spurious mode of instability which arises because the asymptotic formula for Ω used in the calculation predicts $\Omega = 0$ for $ka_e = O(1)$, outside its range of validity. We shall show that the mode of instability for the vortex ring identified in I is analogous to this mode and, therefore, is also spurious.

† We note that for short waves it is not obvious that the resulting motion of the vortex core is just the sum of the induced velocity due to its position in a non-uniform flow plus self-induced rotation in a still fluid. An analysis of this problem for a flow with constant vorticity indicates that the effect of the non-uniform flow enters the boundary conditions at the edge of the core. Thus, the dispersion relation (see, for example, Moore & Saffman 1972) contains an additional term. This appears to have a small effect but should be included in a complete stability calculation for short waves.

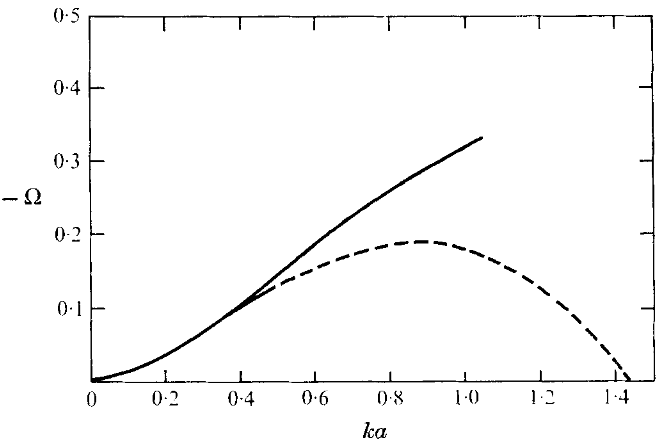

FIGURE 2. Comparison of the exact and asymptotic dispersion relations for waves on a vortex filament with constant vorticity. ——, exact result; – – –, asymptotic ($ka \ll 1$) result.

However, for a straight vortex filament of finite core size, there are other bending modes for which Ω actually does equal zero. These modes have a more complex radial structure with at least one node at some radius within the core. These modes cannot be predicted by the asymptotic long-wave analysis and must be obtained numerically, or in special cases analytically, for each distribution of vorticity within the core from a full solution of the perturbed vorticity equations. We postulate that it is these modes which, having Ω close to zero, will diverge in the stagnation-point flow field.

As an example, we have obtained $\Omega = \Omega(ka)$ for bending waves on a straight filament with constant vorticity by solving numerically for the roots of the dispersion relation given by Moore & Saffman (1972). Figure 3(*a*) shows the result of these calculations. We are interested only in modes for which $\Omega = 0$ for some finite ka. The first radial mode does not have this property (although its asymptotic expression does, as previously discussed). It can be seen in figure 3(*a*) that there are higher radial modes for which Ω passes through zero. The first mode with this property is the second radial mode, for which $\Omega = 0$ at $ka_e = 2{\cdot}5$ ($a_e \equiv a$ for a vortex with constant vorticity).

We have also used an existing computer program (Plobeck 1974) to investigate the dispersion relation for waves on a straight vortex filament with a continuous distribution of vorticity in the core. The distribution chosen was

$$\zeta(r) = (r^2 - a^2)^2 \tag{5}$$

so that $\zeta(r)$ would be continuous at $r = a$. The roots of the dispersion relation obtained from this numerical investigation are shown in figure 3(*b*). The agreement between the asymptotic result and the numerical results was very good for small ka_e. ($a_e = 0{\cdot}7a$ for this particular vorticity distribution.) The numerical results for the second mode predicted $\Omega = 0$ at $ka = 3{\cdot}9$ or at $ka_e = 2{\cdot}7$.

We now wish to re-examine the stability of the vortex ring considering the consequences of including the higher radial bending modes for which Ω can be

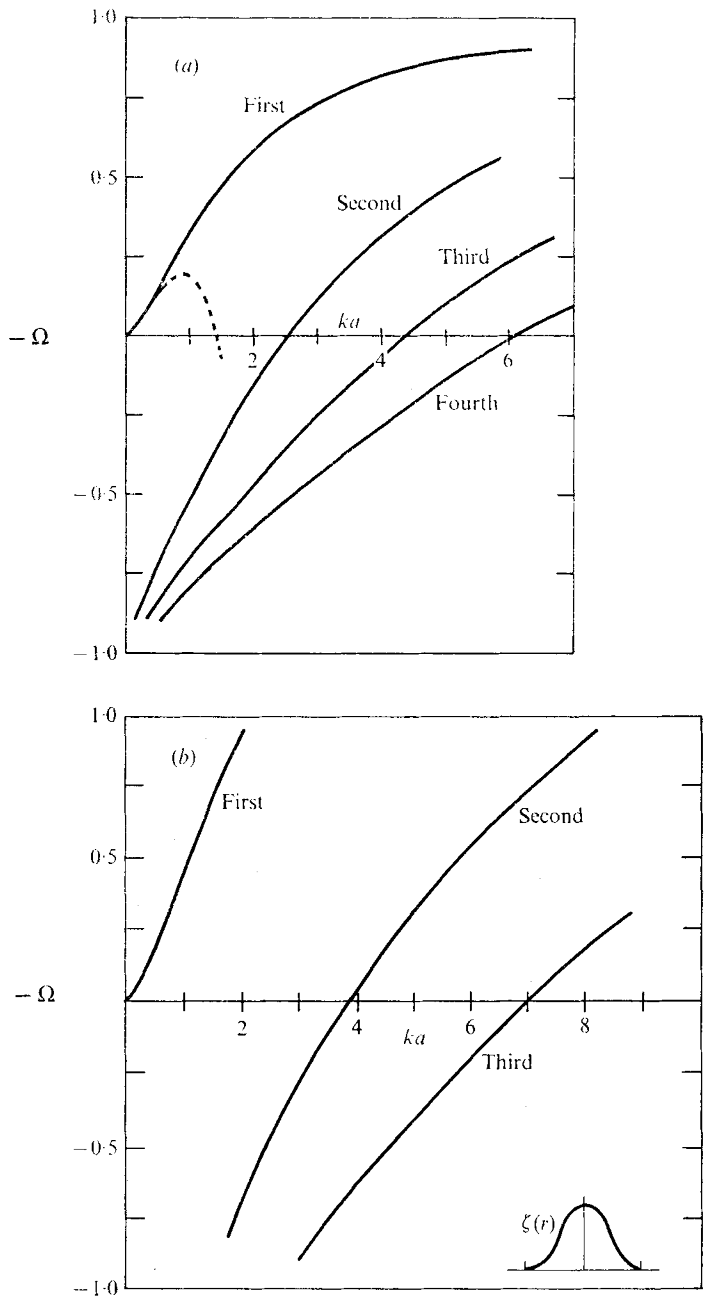

FIGURE 3. Dispersion relations for bending modes on a straight filament. (*a*) With constant vorticity including the higher radial bending modes (second, third, fourth). ——, exact results; – – –, asymptotic ($ka \ll 1$) result for the first mode. (*b*) With distributed vorticity $\zeta(r)$ including the higher radial bending modes. Results obtained by numerical calculation.

zero. We shall also verify that the previous analysis of I is equivalent to using the asymptotic formula for Ω for a value of $ka_e = 1{\cdot}44$, where it spuriously predicts $\Omega = 0$. If we argue that short waves on a slender bent filament will, to lowest order in the limit $a/R \to 0$, $kR \to \infty$, rotate at the same rate as if the filament were straight, we can analyse the vortex-ring instability along the lines for the vortex-pair instability: as a balance between the self-induced rotation due to bending and the flow at the vortex due to the presence of and perturbations on the remainder of the ring. We expect that for short waves, such as are observed on vortex rings, the velocities induced at the core boundary owing to distant perturbations on the ring are negligible; preliminary calculations of the outer potential flow using toroidal co-ordinates indicate that these are of order $1/(kR)^2$ as $kR \to \infty$. (kR can, of course, take only integer values for a ring.)

In a recent study, Bliss (1973) expanded the velocity field near a vortex-filament ring to higher order than was done in II. As in the vortex-pair instability previously discussed, the terms that are of interest for the ring instability are due to the 'stagnation-point' flow induced in the neighbourhood of the vortex core by the presence of the ring. (We again refer to figure 1.) From this analysis, the radial and tangential velocity components of this flow are given by

$$u_r = \frac{\Gamma}{4\pi R^2}\tfrac{3}{4}r \sin 2\theta \left[\ln\frac{8R}{r} - \tfrac{4}{3}\right], \tag{6a}$$

$$u_\theta = \frac{\Gamma}{4\pi R^2}\tfrac{3}{4}r \cos 2\theta \left[\ln\frac{8R}{r} - \tfrac{5}{6}\right]. \tag{6b}$$

We consider that the velocity field (6) for the vortex ring is analogous to the field (1) for the pair. The velocity field (6) is not exactly a stagnation-point flow owing to the presence of the term $\ln r$ and to the different constants in the expressions for u_r and u_θ.

To calculate the actual motion of a curved vortex filament of finite cross-section perturbed with a short-wave disturbance in this non-uniform field requires a detailed asymptotic solution. However, to illustrate what we believe to be the essential mechanism in the vortex-ring instability, we calculate the net translational motion of a cylinder of radius a, representing the vortex core boundary, displaced in this field. To this, we shall add the self-induced rotation due to perturbations on the filament. Because of the presence of the logarithm in the velocity field, to evaluate the net translational velocity we require that the displacement of the vortex core boundary be small in comparison with the core radius.

In the appendix, we show that the net translation of the cylinder of radius a perturbed in the 'stagnation-point' flow (6) is given by [equations (A 5)]

$$u_{r_0} = \frac{\Gamma}{4\pi R^2}\tfrac{3}{4}r_0 \sin 2\theta_0 \left(\ln\frac{8R}{a} - \tfrac{25}{12}\right), \tag{7a}$$

$$u_{\theta_0} = \frac{\Gamma}{4\pi R^2}\tfrac{3}{4}r_0 \cos 2\theta_0 \left(\ln\frac{8R}{a} - \tfrac{25}{12}\right). \tag{7b}$$

These expressions are analogous to the expression in (2) used in the discussion of the vortex-pair instability.

We do not expect that the expressions (7) actually predict the net translational velocity of the core boundary to $O(1)$ since we already know that the same arguments applied to the calculation of the speed of a vortex ring would predict

$$V_0 = \frac{\Gamma}{4\pi R}\left[\ln\frac{8R}{a} - 1\right] \tag{8}$$

rather than the correct expression

$$V_0 = \frac{\Gamma}{4\pi R}\left[\ln\frac{8R}{a_e} - \tfrac{1}{4}\right]. \tag{9}$$

The expression (8) was originally derived by Thompson (1883, p. 33) from reasoning very similar to that presented here while the more complete analysis presented in II or in Saffman (1971) that includes the effects of changes in internal structure of the vortex core gives the expression in (9). However, we are here concerned with the most elementary model of vortex-ring instability which will at least indicate those problems that need to be treated in more detail.

We now suppose that instability of the vortex ring to short-wave perturbations around the azimuth occurs whenever there is a balance between the self-induced rotation Ω of the waves and the net translational velocity (7) induced owing to the ring field. Since the mode of instability considered here has $kR \to \infty$ as $a/R \to 0$, the lowest-order solution gives a value for Ω equal to that for the waves on a straight filament. As in the case of the short-wave instability of the vortex pair, if we compare the asymptotic expression (4) for Ω with the tangential velocity u_{θ_0} given by (7b), we see that it is not possible for u_{θ_0} to balance $r_0\Omega$ as $k \to \infty$ unless the bracket in (4) is close to zero. This occurs only when the asymptotic expression spuriously predicts zero. If Ω could actually equal zero, the rotation would stop at $\theta_0 = 45°$, where $u_{\theta_0} = 0$, and the vortex would diverge at a velocity u_{r_0} given by (7a).

We take as our condition for instability that $\Omega = 0$ for some ka_e and introduce the symbol κ for the value of ka_e at which this occurs. The instability condition $\Omega = 0$ can then be written as

$$k = \kappa/a_e. \tag{10}$$

This can be interpreted as follows: a vortex filament characterized by a value of κ and a_e would be unstable to waves of wavenumber k in the presence of a stagnation-point flow. At this wavenumber, Ω would equal zero and the vortex would diverge. The short-wave instability of both the ring and the pair occur under the conditions given in (10). For a vortex ring, an additional condition must be satisfied: instability can occur only if we can fit an integer number of these waves around the ring, so that

$$k = n/R. \tag{11}$$

We now consider what vortex rings can accommodate integer numbers of these unstable waves.

A slender vortex ring moves with a non-dimensional translation velocity $\tilde{V} = V_0/(\Gamma/4\pi R)$ given by

$$\tilde{V} = \ln(8R/a_e) - \tfrac{1}{4}. \tag{12}$$

The effects of both the core size and vorticity distribution appear in the effective core size a_e. From a combination of (10) and (11), we can see that a ring with a particular value of R/a_e will be unstable to a mode n whenever $R/a_e = n/\kappa$. From (12), this occurs whenever

$$\tilde{V} = \ln(8n/\kappa) - \tfrac{1}{4}. \tag{13}$$

This is the condition that n waves for which $\Omega = 0$ will fit on a particular ring. Because we consider $\Omega = 0$ as the condition for instability rather than balancing the low rotation rates on either side of $\Omega = 0$ with the induced velocity u_{θ_0}, we obtain a discrete set of $\tilde{V}$'s and n's. In reality, a mode n would be unstable for values of $\tilde{V}$ in a band about this value of $\tilde{V}$. The results of I for the value of $\tilde{V}$ for which a given mode n is unstable agree with the predictions of (13) for the value $\kappa = 1{\cdot}44$, the value for the vortex described by the asymptotic formula (4).

As in the vortex-pair instability, if $\Omega = 0$, the vortex will diverge at $\theta_0 = 45°$ with velocity u_{r_0}, given by (7a), corresponding to a non-dimensional amplification rate $\bar{\alpha} = \alpha/(\Gamma/4\pi R)$ given by

$$\bar{\alpha} = \tfrac{3}{4}[\ln(8R/a) - \tfrac{25}{12}] = \tfrac{3}{4}(\tilde{V} - \tfrac{11}{6}). \tag{14}$$

Without a full asymptotic analysis, it is not clear how the core size a in the expression (7a) for u_{r_0} should be chosen. We shall take it to be a_e but recognize that, without further investigation, this is correct only to order $\ln a_e/R$.

For large n, it can be shown that the analysis in I predicts an amplification rate

$$\bar{\alpha} = \tfrac{3}{4}(\tilde{V} - \tfrac{1}{4}). \tag{15}$$

We can see that, as expected, the two approaches differ to $O(1)$. Since (15) is based on a long-wave analysis and the reasoning leading to (14) is known not to be valid to $O(1)$, there is no reason at this stage to attempt to reconcile the two results. The fact that both approaches predict $\bar{\alpha} \sim \tfrac{3}{4}\tilde{V}$ is probably of more significance.

At this point, we have reached the following conclusions.

(i) The essential features of the vortex-ring instability can be understood by considering the balance between the induced velocities due to perturbation in the mean ring field and the self-induced rotation of waves on a straight filament.

(ii) Both the analysis of I and the present analysis predict instability whenever the self-induced rotation is zero and the filament then diverges in the stagnation-point flow due to the ring field.

(iii) The results presented in I are not valid because the asymptotic formula (4) for Ω is not valid at $ka_e = 1{\cdot}44$, at which it predicts zero rotation rate.

But now we consider the other bending waves on the filament which do permit a displacement of the core boundary and yet produce a self-induced rotation that passes through zero (figure 3). These modes have been identified by an analysis that is valid for short waves. They have at least one node in the radial direction. For a particular vortex ring, characterized by $\tilde{V}$, the conditions for vortex-ring instability are again given by (13) with κ now chosen as the value of ka_e for which waves on a straight filament with that particular vorticity distribution would produce no self-induced rotation.

Although in principle one can identify an effective core size for any vortex

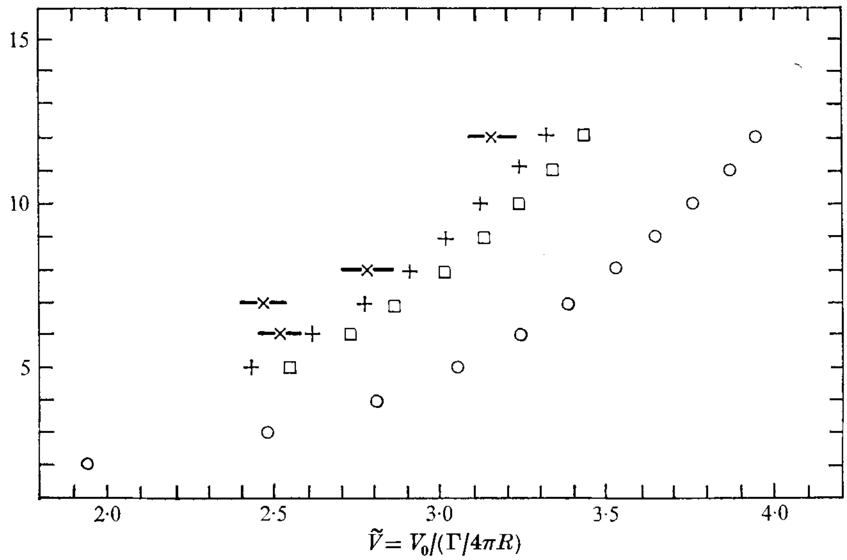

FIGURE 4. Theoretical and experimental results for the value of V for which a given mode n is unstable. ○, asymptotic result, $\kappa = 1{\cdot}44$; □, constant vorticity, $\kappa = 2{\cdot}5$; +, distributed vorticity, $\kappa = 2{\cdot}7$; —×—, experiment, from I.

filament, the concept of kinematic similarity of filaments with equal effective core size does not hold for short waves. Consequently, the value of κ (ka_e for which $\Omega = 0$) would depend upon the details of the vorticity distribution; we obtained $\kappa = 2{\cdot}5$ for a vortex with constant vorticity and $\kappa = 2{\cdot}7$ for the smooth vorticity distribution (5). Since for short waves the dispersion relation and the value of κ depend upon the details of the vorticity distribution, the mode n which is unstable for a given ring ($\tilde{V}$) also depends upon the details of the vorticity distribution. This dependence is rather weak, i.e. as $\ln \kappa$. The value of $\tilde{V}$ for which a given mode n is unstable is shown in figure 4 for vortices with both constant ($\kappa = 2{\cdot}5$) and distributed vorticity ($\kappa = 2{\cdot}7$). Also shown are the results obtained in I, equivalent to using $\kappa = 1{\cdot}44$ from the asymptotic formula (4). The experimental results presented in I are also shown. The agreement between theory and experiment is considerably improved. Instability in the second radial mode for the continuous distribution of vorticity ($\kappa = 2{\cdot}7$) gives particularly close agreement. (The vorticity distribution of one of these rings was measured using LDV techniques by Sullivan, Widnall & Ezekiel (1973); the distribution of vorticity was quite smooth. However, for the simple model presented in this paper, we have not attempted to fit this vorticity exactly to find a more precise value of κ.)

If with the benefit of our new insight (or hindsight) we re-examine the photograph of the instability shown in figure 5 (plate 1), taken from I, we see that the displacement of the centre of the core is, in fact, in the direction opposite to that of displacements of the outer portions of the flow, indicating a second radial mode.

Since $\Omega = 0$, the vortex core diverges along $\theta_0 = 45°$ with the velocity u_{r_0} given by (7a) and the amplification rate $\bar{\alpha}$ is again given by (14); $\bar{\alpha}$ is independent of κ and depends only upon $\tilde{V}$. Since we know that changes in the internal structure of the core due to the perturbation must be included to predict the velocity of translation u_{r_0} to $O(1)$, we do not expect good agreement with the amplification rate obtained in the experiments of I. These particular data were obtained for a rather fat ring ($a/R = 0\cdot3$) with a low value of $\tilde{V}$ ($\tilde{V} = 2\cdot5$) and a moderate number of waves ($n = 7$) in the unstable mode.

Although we feel that the simple model presented here describes the essential features of the vortex-ring instability and that the occurrence of instabilities of higher radial mode on vortex filaments has been demonstrated, there are many questions raised by this analysis that require a more detailed mathematical treatment. One question concerns the dispersion relation for waves on a curved filament. It is clear that for $a/R \to 0$ with $ka \sim O(1)$, so that $kR \to \infty$, the lowest-order solution for the vortex behaviour will correspond to waves on a straight filament, for which we have here presented some results. The effects of curvature enter through the next term in an expansion in a/R. From this, one would conclude that the dispersion relation $\Omega = \Omega(ka_e)$ must pass through $\Omega = 0$ for ka_e somewhere near the value κ for the straight filament. The additional velocities induced by the nearby waves on the curved filament also enter into the analysis at this point.

To calculate the motion of the vortex filament in the non-uniform flow field of the ring, it is necessary to obtain a more complete inner solution for the vortex core. This should include not only the effects of short waves and curvature but also the effects of the displacement of the core in the radial direction, which will change the local curvature and stretch the vortex elements in the core. The inner solution obtained in II cannot be used for short waves.

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Appendix

To calculate the net translational motion of a circular cylinder perturbed in the non-uniform 'stagnation-point' flow of the ring field (6),

$$\left.\begin{aligned} u_r &= \frac{\Gamma}{4\pi R^2}\tfrac{3}{4}r\sin 2\theta\left(\ln\frac{8R}{r}-\tfrac{4}{3}\right),\\ u_\theta &= \frac{\Gamma}{4\pi R^2}\tfrac{3}{4}r\cos 2\theta\left(\ln\frac{8R}{r}-\tfrac{5}{6}\right),\end{aligned}\right\}$$

we refer to the geometry sketched in figure 6. The perturbation is taken to be a displacement r_0 in the direction $\theta = \theta_0$ where $r_0 \ll a$. We calculate the velocity normal $\tilde{u}_r$ to the core boundary in a new cylindrical co-ordinate system (r_1, θ_1) centred at (r_0, θ_0). The 'free-stream' terms which will cause a net translation in the θ_0 direction will be of the form $\tilde{u}_r \sim \cos(\theta_1 - \theta_0)$; terms of the form

$$\tilde{u}_r \sim \sin(\theta_1 - \theta_0)$$

represent translations induced normal to $\theta = \theta_0$. Induced velocities with dependences such as $\sin 2\theta_1$ represent the attempts of the non-uniform flow to

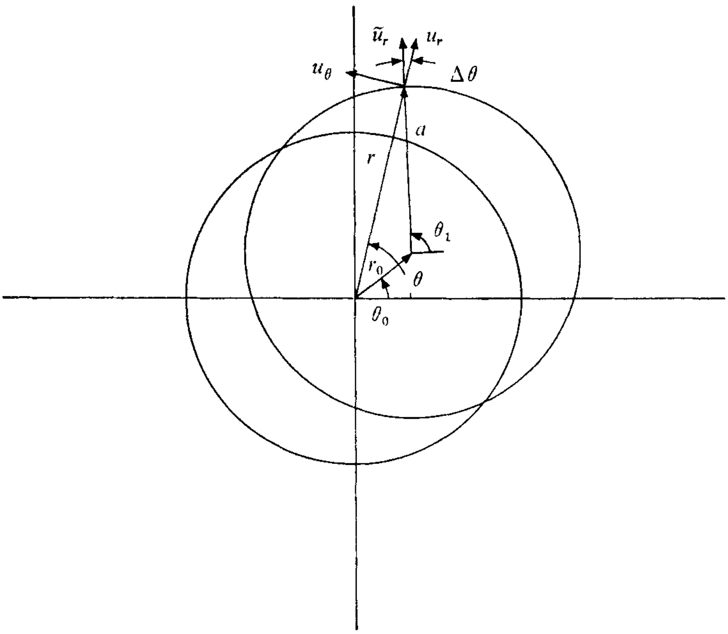

FIGURE 6. Perturbation of the vortex core boundary in the steady non-uniform velocity field of the vortex ring.

change the shape of the vortex core boundary; these terms do not result in any net translation.

From the geometry of figure 6, we can see that the radial velocity $\tilde{u}_r$ is related to the velocities u_r and u_θ through

$$\tilde{u}_r \cong u_r + u_\theta \Delta\theta, \tag{A 1}$$

where $\Delta\theta = \theta_1 - \theta$: the small change in the direction of the normal due to the displacement of the cross-section. For a small displacement r_0, this is given by

$$\Delta\theta \simeq (r_0/a) \sin(\theta - \theta_0). \tag{A 2}$$

The radius to a point on the core boundary is given by

$$r \simeq a + r_0 \cos(\theta - \theta_0). \tag{A 3}$$

Combining (6), (A 1), (A 2) and (A 3) gives the following expression for the radial velocity at the perturbed position of the boundary that would result in a net translation of this boundary (i.e., only the $\sin(\theta_1 - \theta_0)$ and $\cos(\theta_1 - \theta_0)$ terms):

$$\tilde{u}_r = \frac{\Gamma}{4\pi R^2} \tfrac{3}{4} r_0 [\cos 2\theta_0 \sin(\theta_1 - \theta_0) + \sin 2\theta_0 \cos(\theta_1 - \theta_0)] \left(\ln \frac{8R}{a} - \tfrac{25}{12} \right). \tag{A 4}$$

This velocity normal to the core boundary is equivalent to that due to a free stream in the $\theta = \theta_0$ direction of magnitude

$$u_{r_0} = \frac{\Gamma}{4\pi R^2} \tfrac{3}{4} r_0 \sin 2\theta_0 \left(\ln \frac{8R}{a} - \tfrac{25}{12} \right), \tag{A 5a}$$

plus a free stream in the direction normal to $\theta = \theta_0$ of magnitude

$$u_{\theta_0} = \frac{\Gamma}{4\pi R^2} 3r_0 \cos 2\theta_0 \left(\ln \frac{8R}{a} - \tfrac{25}{12}\right). \tag{A 5b}$$

The velocities u_{r_0} and u_{θ_0} in (A 5) are taken to be the net translational velocities in the radial and tangential directions that the core boundary will experience as a result of being displaced in the ring field.

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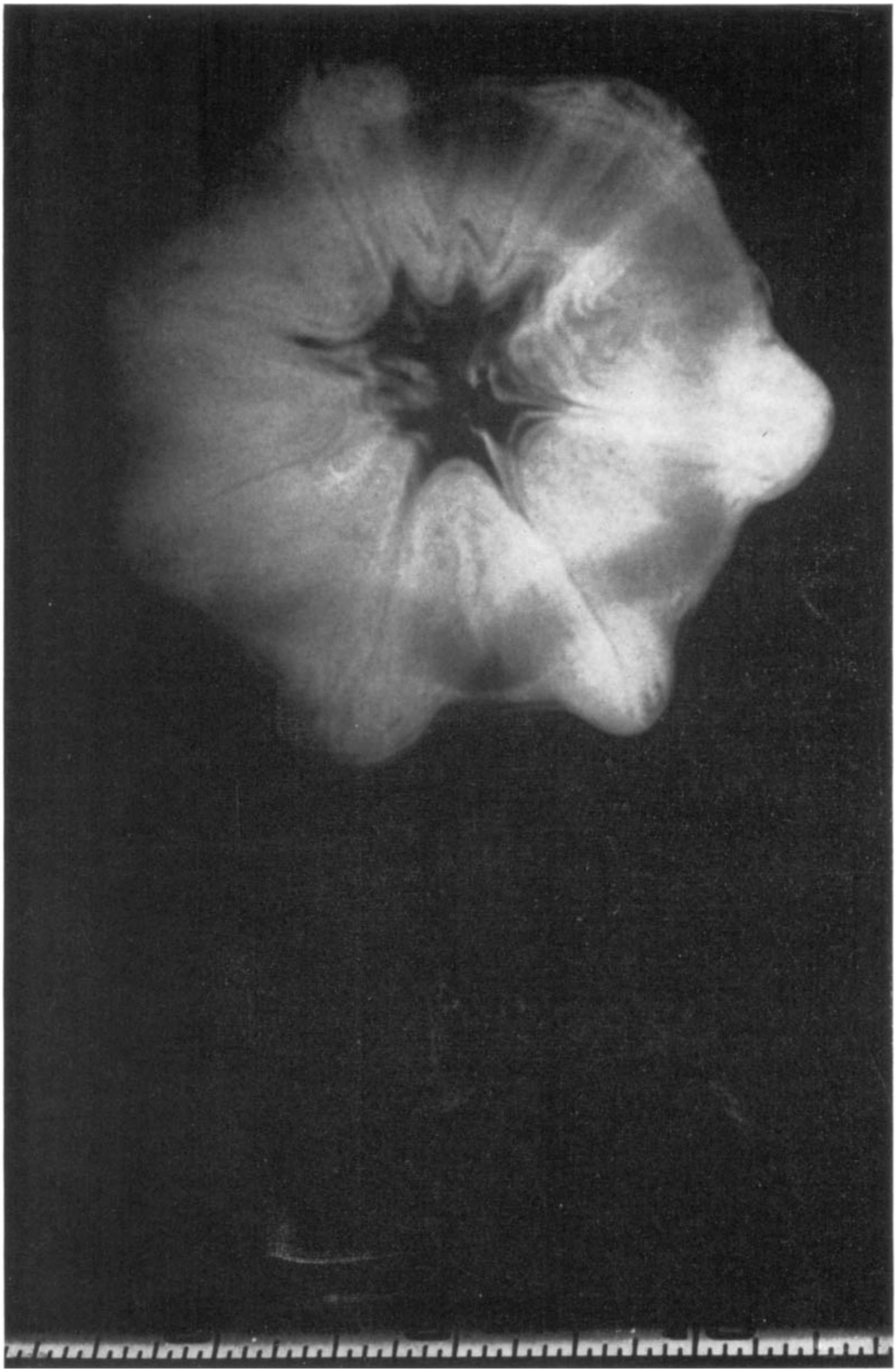

FIGURE 5. Flow visualization of the vortex-ring instability; $n = 7$. Taken from I.